ON THE PROPERTIES OF EQUATIONS OF THE FIRST APPROXIMATION IN THE METHOD OF AVERAGING

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PMM Vol.22, No.5, 1958, pp. 713-719

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(Received 15 January 1958)

The method of averaging which arises in certain problems of celestial mechanics was first applied by Van der Pol [1,2] in problems of the theory of nonlinear vibrations. The method was further developed by Faton [3], Mandel'shtam and Papaleksi [4], Bulgakov [5,6], and Bogoliubov [7].

In the present paper we consider a quasilinear vibrating system, which contains one nonlinear dependence on one of unknown coordinates. The properties of averaged equations of motion are investigated ("shortened", in the first approximation). This work is based on one version of the averaging methods devised by Bulgakov. It is shown that with few assumptions the averaged equations can be reduced to a special form, which allows the establishment of certain properties, and is also useful for a number of specific problems.

1. Let us consider a certain vibrating process of a system with n degrees of freedom defined by the equations

$$\sum_{k=1}^{n} f_{jk}(D)\, y_k = \psi_j(y_l, t) \qquad (j = 1, \ldots, n), \quad \left(D = \frac{d}{dt} \right) \tag{1.1}$$

where y_k are unknown coordinates and $f_{jk}(D)$ is a polynomial with constant coefficients. Only one of the functions ψ_i, say $\psi_m(y_l, t)$ depending on one coordinate y_l and time, can be different from zero. Let $f(D)$ $\| f_{jk}(D) \|$ be a matrix of the system (1.1), $F(D)$ $\| F_{jk}(D) \|$ is the adjoint matrix, such that $F_{kj}(D)$ is the algebraic complement of the elements of $f_{jk}(D)$.

By $\Delta(D) = \det f(D)$ we denote the determinant of the system (1.1), which has θ real roots $\kappa_\sigma (\sigma = 1, \ldots, \theta)$ and conjugate complex roots $\epsilon_h \pm i\omega_h$ $(h = 1, \ldots, \ )$.

We introduce the following assumptions:

(a) Determinant Δ (D) has only simple roots;

(b) Determinant of the coefficients of the highest derivatives in (1.1) differs from zero;

(c) Every coordinate or its derivatives are contained in (1.1).

Using the assumptions above we can transform the system (1.1) to normal coordinates, as done by Bulgakov [6] in the general case.

The formula of transformation has the form:

$$\frac{d^{\nu} y_j}{dt^{\nu}} = \sum_{\sigma=1}^{0} v_{j\sigma}^{\nu} \xi_\sigma + \sum_{h=1}^{\vartheta} N_{jh}{}^{(\nu)} a_h \cos(u_h + \gamma_{jh} + \nu\zeta_h) \qquad (j = 1, \ldots, n, \ \nu = 0, 1, \ldots, m_j - 1) \tag{1.2}$$

Here m_j is the order of highest derivative of coordinate y_j in (1.1), ξ_σ, a_h, u_h are new unknowns (normal coordinates) which satisfy equations

$$\begin{aligned} \frac{d\xi_\sigma}{dt} &= \varkappa_\sigma \xi_\sigma + \frac{w_{\sigma m}}{\Delta'(\varkappa_\sigma)} \psi_m(y_l, t) \\ \frac{da_h}{dt} &= \varepsilon_h a_h + 2 \operatorname{Re}\left[\frac{e^{-iu_h}}{\Delta'(\varepsilon_h + i\omega_h)} W_{\theta+h,m} \psi_m(y_l, t)\right] \\ \frac{du_h}{dt} &= \omega_h + \frac{2}{a_h} \operatorname{Im}\left[\frac{e^{-iu_h}}{\Delta'(\varepsilon_h + i\omega_h)} W_{\theta+h,m} \psi_m(y_l, t)\right] \\ &(\sigma = 1, \ldots, \theta, \qquad h = 1, \ldots, \vartheta) \end{aligned} \tag{1.3}$$

In these equations y_l is replaced with the aid of equation (1.2). The quantities entering into equations (1.2) and (1.3) are defined as

$$v_{j\sigma}^{(\nu)} = v_{j\sigma} \varkappa_\sigma^{\nu}, \qquad N_{jh}^{(\nu)} = N_{jh} \rho_h^{\nu}, \qquad \rho_h e^{i\zeta_h} = \varepsilon_h + i\omega_h$$

$$v_{j\sigma} = s_\sigma \frac{F_{j,l(\sigma)}(\varkappa_\sigma)}{F_{k(\sigma),l(\sigma)}(\varkappa_\sigma)}, \qquad w_{\sigma m} = \frac{1}{s_\sigma} F_{k(\sigma),m}(\varkappa_\sigma)$$

$$N_{jh} e^{i\gamma_{jh}} = s_h \frac{F_{j,l(h)}(\varepsilon_h + i\omega_h)}{F_{k(h),l(h)}(\varepsilon_h + i\omega_h)}, \qquad W_{\theta+h,m} = \frac{1}{s_h} F_{k(h),m}(\varepsilon_h + i\omega_h) \tag{1.4}$$

so that

$$v_{j\sigma} w_{\sigma m} = F_{jm}(\varkappa_\sigma), \qquad N_{jh} e^{i\gamma_{jh}} W_{\theta+h,m} = F_{jm}(\varepsilon_h + i\omega_h)$$

Here s_σ, s_h are arbitrary coefficients, $F_{k(\sigma),l(\sigma)}(\kappa_\sigma)$, $F_{k(h),l(h)}(\epsilon_h + i\omega_h)$ are elements of the matrix $F(D)$ which are not zero for given σ and h.

Equations (1.3) are exact. To obtain simpler approximate equations, we will add new conditions to those previously stated in (a), (b), and (c).

We assume that:

(d) The frequency ω_h is such that the relation $g_1\omega_1 + g_2\omega_2 + \ldots + g\ \omega = 0$ is not fulfilled for any integral values of g_h, which are not all simultaneously zero.

(e) The quantities $\kappa_\sigma\xi_\sigma$, $\epsilon_h a_h$ are small compared with $\epsilon\omega_h a_h$.

(f) The function $\psi_m(y_l, t)$ is small (quasilinear system).

(g) For variations of t, if it enters explicitly, ψ_m varies slowly compared to variation of argument u_h.

With these conditions, equation (1.3) has a "standard" form and permits of averaging with respect to all variables. After averaging we obtain

$$\frac{d\xi_\sigma}{dt} = \varkappa_\sigma\xi_\sigma + \frac{w_{\sigma m}}{\Delta'(\varkappa_\sigma)}\frac{1}{(2\pi)^\vartheta}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\psi_m(y_l,\ t)\,du_1\ldots du_\vartheta \tag{1.5}$$

$$\frac{da_h}{dt} = \varepsilon_h a_h + \frac{2}{(2\pi)^\vartheta}\ \mathrm{Re}\left[\frac{W_{0+h,m}}{\Delta\ (\varepsilon_h + i\omega_h)}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\psi_m e^{-iu_h}du_1\ldots du_\vartheta\right]$$

$$\frac{du_h}{dt} = \omega_h + \frac{2}{a_h(2\pi)^\vartheta}\,\mathrm{Im}\left[\frac{W_{0+h,m}}{\Delta'\,(\varepsilon_h + i\omega_h)}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\psi_m e^{-iu_h}du_1\ldots du_\vartheta\right]$$

$$(\sigma = 1, \ldots, 0, \quad h = 1, \ldots, \vartheta)$$

In averaging, ξ_σ, a_h, t are considered as constants.

The equations of the first two groups do not contain u_h. From the third group u_h can be obtained by quadratures.

2. Assume that any of the quantities $\nu_{l\sigma}(\sigma = 1, \ldots, \theta)$, $N_{lh}(h = 1, \ldots, \ )$ in (1.2) are not zero. Obviously, this restriction is not essential, because if it is not satisfied, the first two equations from the above group can be integrated and the problem becomes trivial. We introduce new unknowns according to the formula

$$x_\sigma = v_{l_\sigma}\xi_\sigma, \qquad z_h = N_{lh}a_h \tag{2.1}$$

Then the expression (1.2) for y_l takes the form

$$y_l = \sum_{\sigma=1}^{0} x_\sigma + \sum_{h=1}^{\vartheta} z_h\cos(u_h + \gamma_{lh}) \tag{2.2}$$

Equations (1.5) and (1.4) lead to

$$\frac{dx_\sigma}{dt} = \varkappa_\sigma x_\sigma + \frac{F_{lm}(\varkappa_\sigma)}{\Delta'(\varkappa_\sigma)} \frac{1}{(2\pi)^\vartheta} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \psi_m(y_l, t)\, du_1 \ldots du_\vartheta$$

$$\frac{dz_h}{dt} = \varepsilon_h z_h + \frac{2}{(2\pi)^\vartheta} \operatorname{Re} A_h, \qquad \frac{du_h}{dt} = \omega_h + \frac{2}{z_h(2\pi)^\vartheta} \operatorname{Im} A_h \tag{2.3}$$

$$(\sigma = 1, \ldots, \theta, \qquad h = 1, \ldots, \vartheta)$$

where

$$A_h = \frac{F_{lm}(\varepsilon_h + i\omega_h)}{\Delta'(\varepsilon_h + i\omega_h)} e^{-i\gamma_{lh}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{-iu_h} \psi_m(y_l, t)\, du_1 \ldots du_\vartheta \tag{2.4}$$

It is easy to show that

$$\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \sin(u_h + \gamma_{lh})\, \psi_m(y_l, t)\, du_1, \ldots, du_\vartheta = 0$$

Consequently,

$$A_h = \frac{F_{lm}(\varepsilon_h + i\omega_h)}{\Delta'(\varepsilon_h + i\omega_h)} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \cos(u_h + \gamma_{lh})\, \psi_m(y_l, t)\, du_1 \ldots du_\vartheta \tag{2.5}$$

Substituting equation (2.5) into equation (2.3) we obtain

$$\frac{dx_\sigma}{dt} = \varkappa_\sigma x_\sigma + \frac{F_{lm}(\varkappa_\sigma)}{\Delta'(\varkappa_\sigma)} \frac{1}{(2\pi)^\vartheta} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \psi_m(y_l, t)\, du_1 \ldots du_\vartheta$$

$$\frac{dz_h}{dt} = \varepsilon_h z_h + \frac{2}{(2\pi)^\vartheta} \operatorname{Re}\left[\frac{F_{lm}(\varepsilon_h + i\omega_h)}{\Delta'(\varepsilon_h + i\omega_h)}\right] \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \cos(u_h + \gamma_{lh})\, \psi_m(y_l, t)\, du_1 \ldots du_\vartheta \tag{2.6}$$

$$\frac{du_h}{dt} = \omega_h + \frac{2}{z_h(2\pi)^\vartheta} \operatorname{Im}\left[\frac{F_{lm}(\varepsilon_h + i\omega_h)}{\Delta'(\varepsilon_h + i\omega_h)}\right] \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \cos(u_h + \gamma_{lh})\, \psi_m(y_l, t)\, du_1 \ldots du_\vartheta$$

$$(\sigma = 1, \ldots, \theta, \quad h = 1, \ldots, \vartheta)$$

Let us consider the function

$$\Phi(x_1, \ldots, x_\theta, z_1, \ldots, z_\vartheta, t) = \frac{1}{(2\pi)^\vartheta} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \left[\int_0^{y_l} \psi_m(y_l, t)\, dy\right] du_1 \ldots du_\vartheta \tag{2.7}$$

which is obtained by the original averaging for the function $\psi_m(y_l, t)$. We will now find the partial derivatives of this function with respect to its argument $x_\sigma (\sigma = 1, \ldots, \theta)$, $z_h (h = 1, \ldots, \ )$. If in a certain domain, the variables x_σ, z_h, u_h satisfy the following conditions, the function

$$\varphi_m(y_l, t) = \int_0^{y_l} \psi_m(y, t)\, dy$$

integrated with respect u_h for any x_σ, z_h, has bounded partial derivatives $\partial\phi_m/\partial x_\sigma$, $\partial\phi_m/\partial z_h$, also integrable with respect to u_h, then the derivatives of the function Φ can be found according to the rule of differentiation of an integral.

Evidently, the indicated condition is fulfilled for a sufficiently large class of functions $\psi_m(y_l, t)$, in particular for stepwise-continuous functions often met with in practice.

Hence, equation (2.6) can be written in the form

$$\frac{dx_\sigma}{dt} = \varkappa_\sigma x_\sigma + \frac{F_{lm}(\varkappa_\sigma)}{\Delta'(\varkappa_\sigma)} \frac{\partial\Phi}{\partial x_\sigma}$$

$$\frac{dz_h}{dt} = \varepsilon_h z_h + 2\mathrm{Re}\left[\frac{F_{lm}(\varepsilon_h + i\omega_h)}{\Delta'(\varepsilon_h + i\omega_h)}\right]\frac{\partial\Phi}{\partial z_h} \qquad (2.8)$$

$$\frac{du_h}{dt} = \omega_h + \frac{2}{z_h}\,\mathrm{Im}\left[\frac{F_{lm}(\varepsilon_h + i\omega_h)}{\Delta'(\varepsilon_h + i\omega_h)}\right]\frac{\partial\Phi}{\partial z_h}$$

$$(\sigma = 1, \ldots, \theta,\ h = 1, \ldots, \vartheta)$$

where the function $\Phi = \Phi(x_1, \ldots, x_\theta, z_1, \ldots, z_\theta, t)$ is given by equation (2.7). For the solution of a number of problems this equation is more convenient than the previous one. However, further transformation of equation (2.8) succeeds in giving another form, which appears useful for the solution of specific problems, but mainly for the investigation of the general properties of the averaged equations.

Assume that both the quantities

$$r_\sigma = \frac{F_{lm}(\varkappa_\sigma)}{\Delta'(\varkappa_\sigma)}, \qquad r_{\theta+h} = 2\mathrm{Re}\left[\frac{F_{lm}(\varepsilon_h + i\omega_h)}{\Delta'(\varepsilon_h + i\omega_h)}\right] \qquad (\sigma = 1, \ldots, \theta, h = 1, \ldots \vartheta)$$

are not zero. If this does not hold, then the problem of investigation of equation (2.8) is trivial, because in that case a part of the equation is integrable, and by substituting the result of integration in the remaining parts, we obtain another form of equation (2.8) with fewer unknowns. Let us introduce the function

$$\Psi(x_1, \ldots, x_\theta, z_1, \ldots, z_\vartheta, t) = \frac{1}{2}\Big[\sum_{\sigma=1}^{\theta} p_\sigma x_\sigma^2 + \sum_{h=1}^{\vartheta} q_\sigma z_h^2\Big] + \Phi(x_1, \ldots, x_\theta, z_1, \ldots z_\vartheta, t) \quad (2.9)$$

where

$$p_\sigma = \frac{\varkappa_\sigma}{r_\sigma}, \qquad q_h = \frac{\varepsilon_h}{r_{\theta+h}}$$

Then, the first two groups of equations (2.8) have the form

$$\frac{dx_\sigma}{dt} = r_\sigma \frac{\partial \Psi}{\partial x_\sigma}, \qquad \frac{dz_h}{dt} = r_{\theta+h} \frac{\partial \Psi}{\partial z_h} \qquad (\sigma = 1, \ldots, \theta, h = 1, \ldots, \vartheta) \quad (2.10)$$

In symmetric notation

$$x_i = x_\sigma, \qquad r_i = r_\sigma \qquad (\sigma = 1, \ldots, \theta, \quad i = 1, \ldots, \theta)$$
$$x_i = z_h, \qquad r_i = r_{\theta+h} \qquad (h = 1, \ldots \vartheta, \ i = \theta + 1, \ldots, \theta + \vartheta)$$

equations (2.10) are transformed into a compact form

$$\frac{dx_i}{dt} = r_i \frac{\partial \Psi}{\partial x_i} \qquad (i = 1, \ldots, \theta + \vartheta) \quad (2.11)$$

which we can, after substitution

$$x_i = x_i' \sqrt{|r_i|}$$

reduce in the same way to the form:

$$\frac{dx_i'}{dt} = \operatorname{sign} r_i \frac{\partial \Psi'}{\partial x_i'} \qquad (i = 1, \ldots \theta + \vartheta) \quad (2.12)$$

where

$$\Psi'(x_1', \ldots, x_{\theta+\vartheta}', t) = \Psi(x_1' \sqrt{|r_1|}, \ldots, x_{\theta+\vartheta}' \sqrt{|r_{\theta+\vartheta}|}, t)$$

3. The form of equation (2.12), which is the result of averaging equations (1.5), permits the establishment of certain properties of the solutions of these equations. The character of the curves of the system (2.12) essentially depends upon the sign of the quantity r_i. The problem of the dependence of the properties of the initial system (1.1) on the signs of these equations can be partially solved by using the following assumptions.

For the signs of r_i to be identical, it is necessary for the integral part of the fraction $DF_{lm}(D)/\Delta(D)$ to differ from zero, or for the fraction itself not to vanish at $D = 0$.

In the sequel we will consider mainly systems with identical signs of r_i, which do not depend explicitly on the time-function Ψ'. In this case equation (2.12) has a form

$$\frac{dx_i}{dt} = \frac{\partial \Psi}{\partial x_i} \qquad (i = 1, \ldots, \theta + \vartheta) \tag{3.1}$$

for $r_i > 0$. The form for $r_i < 0$ is obtained by changing the sign either of the function Ψ or the time t (here the dash sign is omitted, in order not to complicate the notation). The function Ψ in equation (3.1) represents the velocity potential.

The system being considered has been proved correct by a theorem of Barbashin [8]. If a dynamical system possesses a single-valued velocity potential, then every point of the space M for which it is prescribed is either in motion or at rest.

Hence follows the particularly important deduction: among the integral curves of the system (3.1) closed cycles do not exist.

Consider the behavior of the integral curves of the system (3.1) in the neighborhood of a singular point. Let the function Ψ be such that in the neighborhood of the origin it can be expanded in a Maclaurin series

$$\Psi(x_1, \ldots, x_{\theta+\vartheta}) = \Psi(0, \ldots 0) + \sum_{i=1}^{\theta+\vartheta} \Psi_{x_i}(0, \ldots, 0)\, x_i +$$
$$+ \frac{1}{2} \sum_{i,k=1}^{\theta+\vartheta} \Psi''_{x_i x_k}(0, \ldots, 0)\; x_i x_k + o(x_1, \ldots, x_{\theta+\vartheta})$$

Because the point $[0, \ldots, 0]$ is a singular point, $\Psi'_{x_i}(0, \ldots, 0) = 0$, then the motion close to this point is defined by equation

$$\frac{dx_i}{dt} = \sum_{k=1}^{\theta+\vartheta} \Psi''_{x_i x_k}(0, \ldots, 0)\, x_k \qquad (i = 1, \ldots, \theta + \vartheta) \tag{3.2}$$

We assume, as usual, that the determinant of the right-hand side of equation (3.2) differs from zero. Obviously, the roots of the characteristic determinant of the system (3.2) are represented by means of the symmetric matrix $\| \Psi''_{x_i x_k}(0, \ldots, 0) \|$, and consequently are all real. The general solution of equation (3.2) has the form

$$x_i = \sum_{k=1}^{\theta+\vartheta} A_{ik} C_k e^{\lambda_k t} \quad (i = 1, \ldots, \theta + \vartheta)$$

where C_k are arbitrary constants, A_{ik} are constant or polynomials in t and all λ_k are real. Hence it follows that in the case of two variables the singular point is either a nodal or saddle point but not a focus.

It is possible to give a geometrical interpretation for equation (3.1). In fact, it follows that the vector dn with coordinates $dx_1, \ldots, dx_{\theta+}$

is collinear to the vector grad Ψ, but the vector grad Ψ on the surface $\Psi = C$ is orthogonal to the function Ψ. Thus, the trajectories of the system (3.1) are orthogonal to the surface $\Psi = C$. In addition, note that singular points of the system (3.1) defined by the system of the equations $\partial \Psi / \partial x_i = 0$ ($i = 1, \ldots, \theta +$) represent stationary points on the surface Ψ. These circumstances permit the study of trajectories of the system (3.1) to be reduced to the study of the property of the surface Ψ. Now we look for a completely arbitrary function Ψ. By virtue of equation (3.11) it follows that

$$\frac{d\Psi}{dt} = \sum_{i=1}^{\theta+\vartheta} \frac{\partial \Psi}{\partial x_i} \frac{dx_i}{dt} = \sum_{i=1}^{\theta+\vartheta} \left(\frac{\partial \Psi}{\partial x_i}\right)^2 \geqslant 0$$

Therefore, for the motion along the trajectories, the functions ψ can only increase and its derivative is zero only at singular points. The extrema of Ψ are nodal and saddle points - saddle points of system (3.1).

The properties exhibited by the system (3.1) make it possible to propose an approximate method for the construction of the integral curves contained in the construction of the family $\Psi = C$ for different C, and also to construct orthogonal trajectories to that family. It is evident that such a method cannot yield significant accuracy; however, it permits us to obtain a satisfactory qualitative picture of the location of the trajectories in the phase space.

Note one more possibly useful graphic analogy. In case of two variables, considering the surface Ψ (x_1, x_2) in three-dimensional space, it is easy to establish that equation (3.1) determines the projections on the horizontal plane $[x_1, x_2]$ of the material points moving over the surface Ψ (x_1, x_2) under the influence of gravitational and frictional forces.

4. To give an example of the application of the theory, we consider a gyroscopic pendulum, close to the position of equilibrium, under the influence of a small moment caused by the force of dry friction along one of the axes.

The equations of motion in this case have the form [9]:

$$A\ddot{\alpha} + H\dot{\beta} + L\alpha = F \operatorname{sign} \alpha, \qquad B\ddot{\beta} - H\dot{\alpha} + M\beta = 0$$

Denoting

$$\rho = \sqrt{\frac{L}{A}}, \quad \sigma = \sqrt{\frac{M}{B}}, \quad \varkappa = \frac{H}{A}, \quad \lambda = \frac{H}{B}, \quad q = \sqrt{\varkappa\lambda} = \frac{H}{\sqrt{AB}}$$

$$\dot{\alpha} = \Omega, \qquad h = \frac{F}{A}$$

the equations above can be written in the form

$$\dot{\Omega} + \varkappa \dot{\beta} + \rho^2 \alpha = h \operatorname{sign} \Omega, \qquad \ddot{\beta} - \lambda \Omega + \sigma^2 \beta = 0, \qquad \alpha - \Omega = 0 \tag{4.1}$$

The determinant of the system (4.1) has two pairs of imaginary roots:

$$D_{1,2} = \pm i\omega_1, \qquad D_{3,4} = \pm i\omega_2$$

$$\omega_{1,2}^2 = \frac{1}{2}(\rho^2 + \sigma^2 + q^2 \mp f^2), \qquad f^2 = \sqrt{(\rho^2 + \sigma^2 + q^2)^2 - 4\rho^2\sigma^2}$$

Transformation of equation (2.2) for Ω and equation (2.8) in new unknowns have the form

$$\Omega = z_1 \cos(u_1 + \gamma_1) + z_2 \cos(u_2 + \gamma_2)$$

$$\frac{dz_1}{dt} = \frac{\omega_1^2 - \sigma^2}{f^2} \frac{\partial \Phi}{\partial z_1}, \quad \frac{dz_2}{dt} = \frac{\sigma^2 - \omega_2^2}{f^2} \frac{\partial \Phi}{\partial z_2}, \quad \frac{du_1}{dt} = \omega_1, \quad \frac{du_2}{dt} = \omega_2 \tag{4.2}$$

where

$$\Phi(z_1, z_2) = \frac{h}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left[\int_0^{\Omega} \operatorname{sing} \Omega d\Omega \right] du_1 du_2 = \frac{4h}{\pi^2} z_1 [2E(k) - (1 - k^2) K(k)] \tag{4.3}$$

$$k = \frac{z_2}{z_1}, \qquad z_1 \geqslant z_2 \geqslant 0 \quad \big(\Phi(z_1, z_2) = \Phi(z_2, z_1) = \Phi(z_1, -z_2) = \Phi(-z_1, z_2) = \Phi(-z_1, -z_2)\big)$$

$E(k)$, $K(k)$ are complete elliptic integrals. From the last two equations (4.2) it follows that the vibration is isochronic.

Consider the first two equations for the amplitudes. It is easy to show that

$$r_1 = \omega_1^2 - \sigma^2 < 0, \quad r_2 = \sigma^2 - \omega_2^2 < 0$$

Consequently, we have a system for which the velocity potential can be obtained and the deductions of Section 3 are valid. The function Φ in this case is a ruled surface close to the right-hand cone.

For construction of the integral curves the proposed method can be applied. For large H, it is easy to show that $|\omega_1^2 - \sigma^2| \ll |\sigma^2 - \omega_1^2|$; thus, all trajectories are tangent to the z_1-axis, and the frequency of vibration ω_2 (nutation) is damped faster than the frequency ω_1 (precession)

It follows from (4.3) that $\partial\Phi/\partial z_1$, $\partial\Phi/\partial z_2$ are always finite and the vibrations are damped in a finite time. If in equation (4.2) and (4.3) we introduce polar coordinates, then the equations for amplitudes can be evaluated by quadratures. In contrast to [9] the solution is obtained in all the phase plane.

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Translated by M.M.S.